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LETTERS TO THE EDITOR

FUNDAMENTAL FREQUENCY OF THIN ELASTIC PLATES

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Motivated by the work of Jones [1], Sundararajan [2], Laura and his co-authors [3–5], the authors have developed a method to estimate the fundamental frequency of a plate through the finite element solution of its static deflections under a uniformly distributed load without the associated eigenvalue problem. The computed results for the case of a clamped rectangular plate with a central circular hole are found to be in reasonably good agreement with existing test results [6]. The approximate method followed here will be useful for determining the fundamental frequency of elastic plates of arbitrary geometry and boundary conditions.

A frequency–static deflection relation for a thin elastic plate occupying an area A inside the boundary S is [7]

$$\rho h \omega^2 (W_{max}/q) = \alpha_f, \tag{1}$$

where

$$\alpha_f = \iint_A \psi \, dx \, dy \Big/ \iint_A \psi^2 \, dx \, dy. \tag{2}$$

The static deflection of the plate under a uniformly distributed load, q, is taken in the form, $W = W_{max}\psi(x, y)$. W_{max} is the maximum deflection and $|\psi(x, y)| \leqslant 1 \forall (x, y) \in S$. ω is the fundamental frequency, ρ is the mass per unit area and h is the plate thickness.

A four-noded quadrilateral isoparametric plate element is chosen for obtaining the static deflection of a plate under a uniformly distributed load. As in the finite element formulation evaluation of integrals in equation (2) for α_f is described below. The geometry of a four-noded plane isoparametric element is mapped into the normalised square space (ξ, η), $(-1 \leqslant \xi \leqslant 1, -1 \leqslant \eta \leqslant 1)$ through the transformation

$$x_e = \sum_{i=1}^{4} N_i(\xi, \eta) x_{ie}, \qquad y_e = \sum_{i=1}^{4} N_i(\xi, \eta) y_{ie}, \tag{3, 4}$$

$$N_i(\xi, \eta) = (1 + \xi\xi_i)(1 + \eta\eta_i), \tag{5}$$

where ξ_i, $\eta_i = \pm 1$ for the corner nodes. (x_{ie}, y_{ie}) are the nodal co-ordinates of the element. The displacement W_e within the element is interpolated by the same function $N_i(\xi, \eta)$ of equation (5) for the nodal displacement W_{ie} as

$$W_e = \sum_{i=1}^{4} N_i(\xi, \eta) W_{ie}. \tag{6}$$

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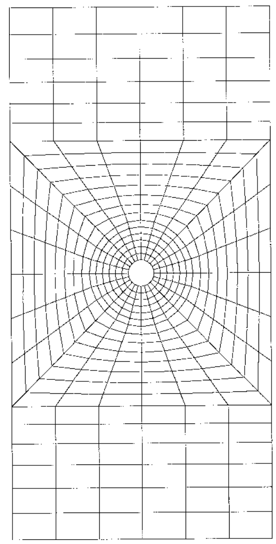

Figure 1. Finite element idealisation for a clamped rectangular plate with a central circular hole.

For a finite element system which is composed of m elements, the integrals in equation (2) can be evaluated to obtain α_f as

$$\alpha_f = W_{max} \sum_{e=1}^{m} \left(\int_{-1}^{1} \int_{-1}^{1} W_e \det |J_e| \, d\xi \, d\eta \right) \Bigg/ \sum_{e=1}^{m} \left(\int_{-1}^{1} \int_{-1}^{1} W_e^2 \det |J_e| \, d\xi \, d\eta \right), \qquad (7)$$

where the determinant of the Jacobian of transformation $[J_e]$ is

$$\det |J| = (\partial x_e/\partial \xi)(\partial y_e/\partial \eta) - (\partial x_e/\partial \eta)(\partial y_e/\partial \xi).$$

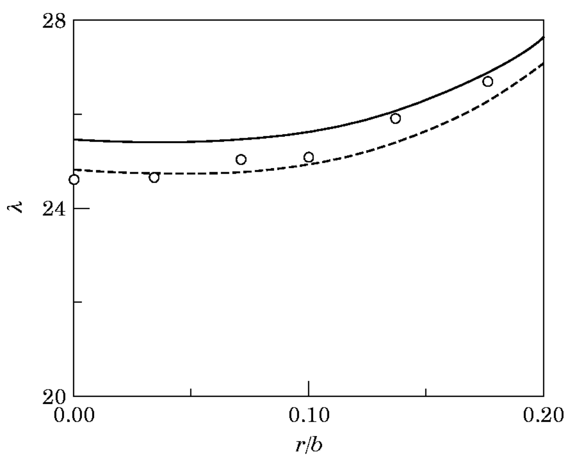

Figure 2. Comparison of the nondimensional frequency parameter, λ with r/b for a clamped rectangular plate with a central circular hole. —, present study; ---, free vibration analysis (NISA); ○, test data [6].

TABLE 1

Comparison of the non-dimensional frequency parameter (λ) for clamped rectangular plate with a central circular hole ($a/b = 2$)

r/b	Free vibration analysis (NISA) (λ)	Present study α ($\times 10^{-2}$)	α_f	λ
0·00	24·81	0·2539	1·6454	25·46
0·05	24·72	0·2669	1·7242	25·40
0·10	24·91	0·2752	1·7994	25·59
0·15	25·60	0·2665	1·8388	26·25
0·20	27·00	0·2455	1·8645	27·56

A ten-point Gaussian integration is adopted to evaluate the integrals. The fundamental frequency, ω is determined from equation (1) after calculating the value of the constant, α_f. The results are presented in the non-dimensional form as

$$W_{max} = \alpha(qb^4/D), \tag{8}$$

$$\lambda = \sqrt{\rho h/D}\omega b^2 = \sqrt{\alpha_f/\alpha} \tag{9}$$

where D is the flexural rigidity.

The adequacy of the present method of computing the fundamental frequency is examined for the cases of a cantilever rectangular plate and a clamped rectangular plate with a central circular hole. A finite element solution is obtained for the static deflections of these plates by using a four-noded quadrilateral isoparametric plate element available in the well-known finite element code NISA (Numerically Integrated elements for System Analysis). Finite element idealisation for a clamped rectangular plate with a central circular hole is shown in Figure 1. a and b are the length and width of a rectangular plate, and r is the radius of the circular hole.

The finite element result for the non-dimensional maximum static deflection, α, of a clamped rectangular plate ($a/b = 2$) is 0·002539 whereas reference [8] gives 0·0025. The results compare well with those of the free vibration analysis using NISA and the test results compiled by Leissa [6], in Figure 2 as well as in Tables 1 and 2.

This approach is also applied to a clamped and simply supported circular plate with a concentric circular, free hole. This case is interesting since an exact solution is available for the frequency [6], and for the static loading problem [8]. Finite element idealisation of the annular plate is shown in Figure 3. The ratio of inner radius (r_i) to outer radius (r_o) is taken as 0·5. The fundamental frequency, ω, is determined from equation (1) after

TABLE 2

Comparison of the non-dimensional frequency parameter (λ) for a cantilever rectangular plate

a/b	Free vibration analysis (NISA) (λ)	Test result, [6] (λ)	Present study α	α_f	λ
0·5	3·4969	3·34	0·1278	1·5753	3·5106
1·0	3·4759	3·37	0·1291	1·5715	3·4895
2·0	3·4410	3·36	0·1309	1·5623	3·4549

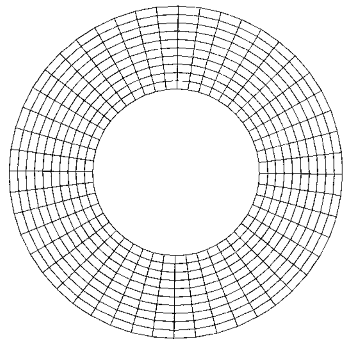

Figure 3. Finite element idealisation for a circular plate with a concentric circular hole.

TABLE 3

Comparison of the non-dimensional frequency parameter (λ) *for a clamped and simply supported circular plate with a concentric circular free hole* ($r_i/r_0 = 0{\cdot}5$), $\nu = 1/3$

Case	Roark [8] α	Free vibration analysis [NISA] λ	Exact solution [6] λ	Present study α		
				α	α_f	λ
Simply supported	0·0624	5·0116	5·040	0·0624	1·6116	5·019
Clamped	0·0053	17·3830	17·510	0·0053	1·6796	17·530

calculating the value of the constant, α_f. The results in Table 3 are presented in the non-dimensional form as

$$W_{max} = \alpha(qr_o^4/D) \tag{10}$$

$$\lambda = \sqrt{\rho h/D}\omega r_o^2 = \sqrt{\alpha_f/\alpha} \tag{11}$$

The results compare well with those of the free vibration analysis using NISA and the exact solution given in reference [6].

The present method of computing the fundamental frequency is found to be in reasonably good agreement with the available exact solution/test results of thin elastic plates.

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